



ELSEVIER

Topology and its Applications 125 (2002) 215–231

TOPOLOGY
AND ITS
APPLICATIONS

www.elsevier.com/locate/topol

Which spaces have a coarser connected Hausdorff topology?

William Fleissner, Jack Porter, Judith Roitman

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

Received 4 July 2000; received in revised form 2 October 2001

Abstract

We present some answers to the title. For example, if K is compact, zero-dimensional and D is discrete, then $K \oplus D$ has a coarser connected topology iff $w(K) \leq 2^{|D|}$. Similar theorems hold for ordinal spaces and spaces $K \oplus D$ where K is compact, not necessarily zero-dimensional. Every infinite cardinal has a coarser connected Hausdorff topology; so do Kunen lines, Ostaszewski spaces, and Ψ -spaces; but spaces X with $X \subset \beta\omega$ and $|\beta\omega \setminus X| < 2^c$ do not. The statement “every locally countable, locally compact extension of ω with cardinality ω_1 has a coarser connected topology” is consistent with and independent of ZFC. If X is a Hausdorff space and $w(X) \leq 2^\kappa$, then X can be embedded in a Hausdorff space of density κ .

© 2001 Elsevier Science B.V. All rights reserved.

MSC: 54C10; 54D05; 54D35

Keywords: Connected; Condense; Ordinal; Coarser topology; Extension; Minimal Hausdorff; Franklin–Rajagopalan space

1. Goals

All spaces in this paper are assumed to be Hausdorff. All ordinals (hence all cardinals) are assumed to have the order topology. For a space X , let $\tau(X)$ denote the collection of open sets of X . A Hausdorff topology $\sigma \subseteq \tau(X)$ is called a coarser topology.

We continue the quest begun in [10]—that is, we seek necessary and sufficient conditions that a space X have a coarser connected topology. (Russians would say “ X condenses onto a connected Hausdorff space”.) There probably is no nice answer for all spaces, but for certain classes of spaces, there are nice answers. Here is a trivial example. A minimal Hausdorff space (a fortiori, a compact space) has a coarser connected topology iff it already is connected.

We are interested in more substantial examples, such as the following from [8].

E-mail addresses: fleissne@math.ukans.edu (W. Fleissner), porter@math.ukans.edu (J. Porter), roitman@math.ukans.edu (J. Roitman).

0166-8641/01/\$ – see front matter © 2001 Elsevier Science B.V. All rights reserved.

PII: S0166-8641(01)00274-7

Theorem 1. *Let X be a not connected space with a countable network. X has a coarser connected topology iff X is not H -closed.*

The first part of this article aims to prove theorems like

Theorem 2.

- (a) *Let $X = K \oplus D$ where D is discrete. If $w(K) \leq 2^{|D|}$ then X has a coarser connected topology.*
- (b) *Let $X = K \oplus D$ where D is discrete and K is minimal Hausdorff with a π -base of clopen sets.¹ If X has a coarser connected topology then $w(K) \leq 2^{|D|}$.*

A similar result is a necessary and sufficient condition for an ordinal to have a coarser connected topology, a particular case of which is

Theorem 3. *Every cardinal has a coarser connected topology.*

In Section 2, we prove Theorem 2(b) and similar theorems. In Section 3, we prove that if $w(K) \leq 2^\kappa$, then K can be embedded in a space of density κ ; this result is needed to prove Theorem 2(a). In Section 4, we introduce the notion “epoxic”, and use it to prove Theorem 2(a) and similar theorems. In Section 5 we introduce the notion of “super-epoxic”, and use it to prove Theorem 3 and similar theorems.

Because of the central role played in Sections 4 and 5 by spaces with dense discrete subspaces, we investigate which extensions of ω have coarser connected topologies. In Section 6, we consider locally compact, locally countable extensions of ω , and in Section 7, we consider spaces X satisfying $\omega \subseteq X \subseteq \beta\omega$.

2. Necessary conditions

We rely heavily on the notions of minimal Hausdorff, H -closed, and semiregular spaces, where a space is *minimal Hausdorff* iff it has no proper subtopology (recall that “space” means “Hausdorff space”), a space X is *H -closed* iff for every \mathcal{C} an open cover of X there is a finite subfamily \mathcal{D} with $X = \text{cl}_X(\bigcup \mathcal{D})$, and a space is *semiregular* iff its regular-open sets form a base. The regular-open sets are defined to be $\mathcal{RO}(X) = \{\text{int}_X \text{cl}_X U : U \in \tau(X)\}$. We say that Y is an *extension* of X iff X is a dense subspace of Y .

We remind the reader of the following facts: A space is minimal Hausdorff iff it is H -closed and semiregular; a compact space is minimal Hausdorff; every space can be densely embedded in an H -closed space.

For a space X , let $X(s)$ denote X with the topology generated by $\mathcal{RO}(X)$. The space $X(s)$ is semiregular and Hausdorff. Also, note that semiregularity is hereditary on dense subspaces and open sets but not (in general) on regular-closed (the closure of an open set) sets. These results and more can be found in [9].

¹ E.g., K is compact and zero-dimensional.

Lemma 2.1.

- (a) If X is connected, so is $X(s)$.
- (b) If Y is a semiregular extension of X , then $wY \leq |\tau(X)|$.
- (c) Let X be a semiregular space and $U \in \tau(X)$ such that $B = \text{cl}_X U \setminus U$ is compact. Then $Z = \text{cl}_X U$ is semiregular.

Proof. The proof of (a) is straightforward. To prove (b), first note that $\{\text{int}_Y \text{cl}_Y U : U \in \tau(Y)\} = \{\text{int}_Y \text{cl}_Y (U \cap X) : U \in \tau(Y)\} = \{\text{int}_Y \text{cl}_Y V : V \in \tau(X)\}$. Thus, $wY \leq |\tau(X)|$.

For part (c), let $p \in V \in \tau(Z)$. We may assume that $V = V' \cap Z$ where $V' = \text{int}_X \text{cl}_X V$. Because $B \setminus V'$ is compact, there is some $T \in \tau(X)$ such that $p \in T \subseteq V'$ and $\text{cl}_X T \cap B \setminus V' = \emptyset$. It suffices to show that $\text{int}_Z \text{cl}_Z (T \cap Z) \subseteq V$. Note that $\text{int}_Z \text{cl}_Z (T \cap Z) \cap B \subseteq \text{cl}_X T \cap B \subseteq V$, that $S = \text{int}_Z \text{cl}_Z (T \cap Z) \cap U \in \tau(X)$ and that $S \subseteq \text{cl}_X T \subseteq \text{cl}_X V$. Thus, $S = \text{int}_X S \subseteq \text{int}_X \text{cl}_X V = V$. So, $\text{int}_Z \text{cl}_Z (T \cap Z) \subseteq S \cup (\text{int}_Z \text{cl}_Z (T \cap Z) \cap B) \subseteq V$. \square

Lemma 2.2. Let X be a minimal Hausdorff space with a π -base of clopen sets.

- (a) If X is a subspace of a semiregular connected space (Y, σ) , then $wX \leq |\sigma(Y \setminus X)|$.
- (b) Hence, if X is a subspace of a space (Y, τ) which has a coarser connected topology σ , then $wX \leq |\tau(Y \setminus X)|$.

Proof. For part (a), it suffices (by Lemma 2.1(b)) to show that $Y \setminus X$ is dense in Y . Towards a contradiction, assume that U is a nonempty, open (in Y) subset of X . Let V be nonempty, clopen in X such that $V \subseteq U$. First note that $V = V \cap U$ is open in Y . Next note that V , closed in X , which is H -closed, is closed in Y . Thus V is a nontrivial clopen subset of Y , a contradiction.

For part (b), we may assume (via Lemma 2.1(a)) that (Y, σ) is semiregular. Then $w(X, \tau) = w(X, \sigma) \leq |\sigma(Y \setminus X)| \leq |\tau(Y \setminus X)|$. \square

Corollary 2.3 (= Theorem 2(b)). Let X be a minimal Hausdorff space with a π -base of clopen sets, D a discrete space. If $X \oplus D$ has a coarser connected topology, then $w(X) \leq 2^{|D|}$.

The converse of Corollary 2.3 will follow from Corollary 5.5.

Another corollary to Lemma 2.2 concerns ordinal sums.

Corollary 2.4. Let δ be a limit ordinal. Set $\beta = \inf\{\gamma > 0 : \exists \eta \ \delta = \eta + \gamma\}$ and set $\alpha = \inf\{\gamma : \gamma + \beta = \delta\}$. If δ has a coarser connected topology then $|\alpha| \leq 2^{|\beta|}$.

Proof. δ is homeomorphic to $(\alpha + 1) \oplus \beta$ and $w(\alpha + 1) = |\alpha|$. \square

For example, $(2^\omega)^+ + \omega$ has no coarser connected topology. We will show in Section 5 that it is the least limit ordinal with no coarser connected topology.

Note that if δ is a cardinal, then $\alpha = 0$ and $\beta = \delta$. Because successor ordinals are compact spaces, the only successor ordinal with a coarser Hausdorff topology is 1. Theorem 5.7 is the converse of Corollary 2.4.

The following concept allows us to prove an analog of Corollary 2.3 for compact spaces, whether or not they have a π -base of clopen sets.

Definition 2.5. Let X be a space. (a) A quasi-component of X is the intersection of a maximal filter of clopen sets.² (b) A subset $A \subseteq X$ is a *qc-transversal* if for each quasi-component C of X , $|A \cap C| = 1$.

Lemma 2.6. Let X be a compact space and D a discrete space. If $X \oplus D$ has a coarser connected topology, then there is a qc-transversal $H \subseteq X$ such that $|w(\text{cl}_X H)| \leq 2^{|D|}$.

Proof. Suppose $X \oplus D$ has a coarser connected topology, say σ . Let $Y = X + D$, the disjoint union of X and D , with the topology σ . By Lemma 2.1(a), we can assume that Y is semiregular. If B is a clopen subset of X , then $B \cap \text{cl}_Y D \neq \emptyset$. So if C is a quasi-component of X , then $C \cap \text{cl}_Y D \neq \emptyset$. There is a qc-transversal $H \subseteq X$ such that $\text{cl}_X H \subseteq \text{cl}_Y D$. Now, $\text{cl}_Y D \setminus D = \text{cl}_Y D \cap X$ is compact. By Lemma 2.1(c), $\text{cl}_Y D$ is semiregular. Since $\tau(X) = \sigma|_X$ as X is minimal Hausdorff, $w(\text{cl}_X H) = w(\text{cl}_X H, \sigma|_{\text{cl}_X H}) \leq w(\text{cl}_Y D, \sigma|_{\text{cl}_Y D})$. By Lemma 2.1(b), $w(\text{cl}_Y D) \leq |\sigma|_D| \leq |\tau(D)| = 2^{|D|}$. \square

Theorem 5.8 is the converse of Lemma 2.6.

3. Embeddings, weight, and density

We aim for Theorem 3.3, which is needed for the converse of Corollary 2.3. We begin with a theorem of Magill [7] (also see [4, Theorem 3.5.13]). We also show that Corollary 2.3 cannot be strengthened in what might appear to be natural ways.

Theorem 3.1. Let X be a locally compact space and Y a compact space such that Y is the continuous image of $\beta X \setminus X$. Then there is a compactification Z of X such that $Z \setminus X$ is homeomorphic to Y .

Lemma 3.2. Let K be a compact space such that $d(K) \leq \kappa$ and D be a discrete space of cardinality κ . Then there is a compactification cD of D such that $cD \setminus D$ is homeomorphic to K .

Proof. Let $f : D \rightarrow K$ be a function such that $f[D]$ is dense in K and $|f^\leftarrow[f(d)]| = \omega$ for every $d \in D$. The continuous extension $\beta f : \beta D \rightarrow K$ is onto. Moreover, since $|f^\leftarrow[f(d)]| = \omega$ for $d \in D$, $\beta f|_{\beta D \setminus D} : \beta D \setminus D \rightarrow K$ is onto. By Theorem 3.1, there is a compactification cD of D such that $cD \setminus D$ is homeomorphic to K . \square

Theorem 3.3. Let X be a space such that $w(X) \leq 2^\kappa$ for some cardinal κ , and let D be a discrete space with $|D| = \kappa$. There is an extension hD of D such that $hD \setminus D$ is

² Quasi-components are the same as components in compact spaces.

homeomorphic to X . Hence every space of weight at most 2^κ can be embedded in a space of density κ .

Proof. This result is straightforward when X is Tychonoff. Let $e: X \rightarrow K$ be an embedding, where K is the product space \mathbf{I}^{2^κ} ($\mathbf{I} = [0, 1]$) which has density κ . Then we apply Lemma 3.2 to K . Finally, let $hD = D \cup e[X]$.

If X is Hausdorff but not Tychonoff, we obtain an H-closed extension Y of X such that $wX = wY$ via a result of Flachsmeier [5, Cor. 1.1]. Let \mathcal{B} be a base for Y such that $|\mathcal{B}| \leq 2^\kappa$. Let \mathcal{C} be the Boolean subalgebra of $\mathcal{RO}(Y)$ generated by

$$\{\text{int}_Y \text{cl}_Y U : U \in \mathcal{B}\} \cup \{Y \setminus \text{cl}_Y U : U \in \mathcal{B}\}.$$

By [9, 3.1(1)], $|\mathcal{B}| = |\mathcal{C}|$. Let EC be the Stone space generated by \mathcal{C} . The space EC is a compact 0-dimensional space and $w(EC) = |\mathcal{C}| = |\mathcal{B}|$. Also, the function $\phi: EC \rightarrow Y$ defined by $\phi(\mathcal{U}) = \bigcap \{\text{cl}_Y U : U \in \mathcal{U}\}$ is perfect (i.e., closed and point-inverses are compact) and onto (because Y is H-closed) but not necessarily continuous (see [5]). Let Z be the compact 0-dimensional product space 2^{2^κ} where $2 = \{0, 1\}$ with the discrete topology. So, $dZ \leq \kappa$ and EC is a subspace of Z . By Lemma 3.2, there is a compactification cD of D such that $cD \setminus D$ is homeomorphic to Z . Consider the function θ from the subspace $D \cup EC$ of cD to $D \cup Y$ defined by $\theta(d) = d$ for each $d \in D$ and $\theta(y) = \phi(y)$ for each $y \in EC$. Let σ be the topology on $D \cup Y$ generated by the base $\{Y \setminus \theta[A] : A \text{ is closed in } EC\}$. It is straightforward to show that D is discrete and dense in $D \cup Y$ and that $D \cup Y$ is Hausdorff. Let A be a closed subset of $D \cup EC$. Then $\theta[A] = \phi[A \setminus D] \cup (A \cap D)$. Since ϕ is closed, $\phi[A \setminus D]$ is closed in $(Y, \sigma|_Y)$. Thus, $\sigma|_Y \subseteq \tau(Y)$. By [DP, 0.1], there is a Hausdorff topology ρ on $D \cup Y$ such that D is discrete and dense in $(D \cup Y, \rho)$ and $\rho|_Y = \tau(Y)$. \square

It is natural to ask whether in Corollary 2.3 it is possible to replace “ $w(X) \leq 2^{|D|}$ ” with “ $|\{A \subseteq X : A \text{ clopen}\}| \leq 2^{|D|}$ ”. The next example, an application of Lemma 3.2, shows that the answer is no.

Example 3.4. Let D be a discrete space such that $|D| = \mathfrak{c} = 2^\omega$. We construct a space Z with D dense in Z so $Z \oplus \omega$ has no coarser connected topology.

The product space $\mathbf{I}^{2^\mathfrak{c}}$ is a connected compact space such that $d(\mathbf{I}^{2^\mathfrak{c}}) \leq \mathfrak{c}$ and $w(\mathbf{I}^{2^\mathfrak{c}}) = 2^\mathfrak{c}$. By Lemma 3.2 there is a compactification Z of D such that $Z \setminus D$ is homeomorphic to $\mathbf{I}^{2^\mathfrak{c}}$. The only clopen sets of Z are the finite subsets of D and their complements. That is, Z has a π -base of clopen sets and $|\{A \subseteq Z : A \text{ clopen}\}| = \mathfrak{c}$. If $Z \oplus \omega$ has a coarser connected topology, then $wZ \leq 2^\omega$ by Lemma 2.3. Then $w(\mathbf{I}^{2^\mathfrak{c}}) = w(Z \setminus D) \leq wZ \leq \mathfrak{c}$. But $w(\mathbf{I}^{2^\mathfrak{c}}) = 2^\mathfrak{c}$, a contradiction.

It is also natural to ask whether in Corollary 2.3 it is possible to omit the hypothesis that X has a π -base of clopen sets. Citing an example from the next section, the answer is no.

Example 3.5. Let X be compact connected. We show that $X \oplus \omega$ has a coarser connected topology.

From Example 4.1, there is a countable connected space E . Let p be a point in X , q a point in E . Take the quotient topology on $X \oplus E$ that identifies p and q . This is a coarser connected topology of $X \oplus (E \setminus \{q\})$ which in turn is a coarser topology on $X \oplus \omega$.

4. Sufficient conditions via epoxicity

While Urysohn [11] constructed the first example of a countable, connected (Hausdorff) space, Bing's "sticky foot" space is a simpler example. The reader is referred to [1] and [4, Example 6.1.6] for a geometric presentation. Here we are interested not only in the space, but also in its extensions and bases. We construct it as a dense subspace of a quotient space, so that it is a paradigm for the notion of ω -epoxic (Definition 4.3). Epoxicity is the main tool we will use to show that spaces of the form $Z \oplus D$ have coarser connected topologies.

Example 4.1 (The "sticky foot" space). Let $X = \{q + q'\sqrt{3} : q, q' \in \mathbb{Q} \text{ and } q' \geq 0\}$, $R = \{q - q'\sqrt{3} : q, q' \in \mathbb{Q} \text{ and } q' \geq 0\}$, $T = \{(n+1)\sqrt{5} : n \in \omega\}$.

Let $E = X \cup R \cup T$, $E' = X \cup T$. We will construct a topology on E' in which X is Bing's sticky foot space, and E' is an extension of X . The topology will be constructed via a quotient map from E onto E' .

First we enlarge the usual subspace topology on E to a topology τ in which $E \setminus \mathbb{Q}$ is closed discrete. For $e \in E$, a basic open neighborhood of e is $(a, b) \cap \mathbb{Q} \cup \{e\}$ where $a < e < b$.

Now we define the map $\varphi : E \rightarrow E'$ by $\varphi(q - q'\sqrt{3}) = q + q'\sqrt{3}$ for $q - q'\sqrt{3} \in R$ and $\varphi(e) = e$ for $e \in E'$.

The topology σ is the quotient topology on the subspace E' (of (E, τ)) determined by φ . Bing's space is the subspace X of (E', σ) .

Let's list some properties of X .

- (1) \mathbb{Q} is dense and open in X .
- (2) A basic open neighborhood of $q \in \mathbb{Q}$ is $(a, b) \cap \mathbb{Q}$ where $a < q < b$.
- (3) A basic open neighborhood of $q + q'\sqrt{3} \in X \setminus \mathbb{Q}$ is $\{q + q'\sqrt{3}\} \cup ((a, b) \cap \mathbb{Q}) \cup ((a', b') \cap \mathbb{Q})$ where $a < q - q'\sqrt{3} < b$ and $a' < q + q'\sqrt{3} < b'$. (These two rational open intervals are the "feet".)
- (4) If $a < q - q'\sqrt{3} < b$ or $a < q + q'\sqrt{3} < b$ then $q + q'\sqrt{3} \in \text{cl}_\sigma((a, b) \cap \mathbb{Q})$. So rational open intervals have large closures (i.e., the feet are sticky).
- (5) If $[a, b] \cap T = \emptyset$ then $\text{cl}_\sigma((a, b) \cap \mathbb{Q}) \cap T = \emptyset$.

To show X is connected, assume that G is a nontrivial clopen subset of X . Then there are a, b, a', b' so that $((a, b) \cap \mathbb{Q}) \subset G$ and $((a', b') \cap \mathbb{Q}) \subset X \setminus G$. Without loss of generality $a < b < a' < b'$. There are rationals q, q' with $a < q - q'\sqrt{3} < b$ and $a' < q + q'\sqrt{3} < b'$. So $q + q'\sqrt{3} \in \text{cl}_X G \cap \text{cl}_X X \setminus G$, a contradiction.

Since X is connected and dense in E' , E' is connected.

From property (5) we obtain a useful property which will be used in Lemma 4.4: Although X is a dense subset of E' , there is a base \mathcal{B} (implicitly defined in (2) and (3) above) for X such that $\text{cl}_{E'} B \subset X$ for all $B \in \mathcal{B}$.

Our next task is to generalize Example 4.1. First we need a definition.

Definition 4.2. We say that a family \mathcal{L} of sets is *linked* if $L \cap L' \neq \emptyset$ for all $L, L' \in \mathcal{L}$. A space X is *Urysohn* iff for all $x, y \in X$ there are open U, V with $x \in U$, $y \in V$, and $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. A space X is *nowhere Urysohn* iff the family of closures of nonempty open sets is linked. Note that if D is dense in X and D is connected (or nowhere Urysohn) then so is X .

Clearly, nowhere Urysohn spaces are connected. Regular spaces are Urysohn, and X is both Urysohn and nowhere Urysohn iff X has only one point. Bing's sticky foot space is nowhere Urysohn.

We call the spaces we use as glue in our constructions *epoxic*.

Definition 4.3. We say that a space X is κ -*epoxic* iff it has an extension E satisfying

- (1) $|E \setminus X| = \kappa$;
- (2) $E \setminus X$ is closed and discrete in E ;
- (3) There is a base \mathcal{B} for X such that $\text{cl}_E B \subset X$ for all $B \in \mathcal{B}$;
- (4) For all disjoint subsets A, A' of $E \setminus X$ there are disjoint open subsets U, U' of E with $A \subset U$ and $A' \subset U'$.

Observe that X is dense in E (by the definition of extension) and open in E (because $E \setminus X$ is closed). If X is locally compact or locally H-closed, then condition (3) is satisfied. Note that if $\kappa' < \kappa$ and X is κ -epoxic, then X is κ' -epoxic.

In Example 4.1, we constructed not only Bing's space X but also an extension E' which demonstrates that X is ω -epoxic.

The next lemma shows how nowhere Urysohn epoxic spaces can be used to create nowhere Urysohn coarser topologies of topological sums.

Lemma 4.4. *The following are equivalent:*

- (1) (X, τ) is κ -epoxic.
- (2) If Z is a space of density κ , then there is a coarser topology on $Z \oplus X$ of which (X, τ) is a dense open subspace.
- (3) If T is the discrete space of cardinality κ , then there is a coarser topology on $\beta T \oplus X$ of which (X, τ) is a dense open subspace.

Proof. (1) \Rightarrow (2): Let Y be dense in Z , $|Y| = \kappa$, let E witness that X is κ -epoxic. Let $\varphi: (E \setminus X) \rightarrow Y$ be bijective. Let σ be the quotient topology determined by φ . Let $Z + X$ denote the space $(Z \cup X, \sigma)$. Note that X is open and dense in $Z + X$.

We show that $Z + X$ is Hausdorff by considering three cases. If $x, x' \in X$ use that X is open in $Z + X$. If $x \in X$ and $z \in Z$, from condition (3) of Definition 4.3, there is an open B with $x \in B$ and $z \notin \text{cl}_\sigma B$. If $z, z' \in Z$, let V, V' be disjoint open subsets of Z with $z \in V$ and $z' \in V'$. Apply condition (4) of Definition 4.3 to $A = \varphi^{-1}(V \cap Y)$ and $A' = \varphi^{-1}(V' \cap Y)$ to get disjoint open subsets U, U' of E . Then $\varphi(U \cup V)$ and $\varphi(U' \cup V')$ are the required disjoint open sets.

(2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1): Let $(\beta T \oplus X, \sigma)$ be the hypothesized coarser topology. Set $E = X \cup T$ (here, and in similar situations, we choose T disjoint from X). We claim that the extension (E, σ) witnesses that X is κ -epoxic. Conditions (1) and (2) are obvious. Recall the fact that in a Hausdorff space, disjoint compact sets can be separated by disjoint open sets. For condition (3), apply the recalled fact to $\{x\}$ and βT ; for condition (4), apply the recalled fact to $\text{cl}_{\beta T} A$ and $\text{cl}_{\beta T} A'$. \square

Corollary 4.5. *Suppose X is κ -epoxic, $d(Z) = \kappa$, and $Z' \subseteq Z$. If X is connected (nowhere Urysohn), then $Z' \oplus X$ has a coarser topology which is connected (nowhere Urysohn).*

Corollary 4.6. *If X is a space with a denumerable closed discrete set W of isolated points and X can be embedded into a separable space Z (for example, X could be an ordinal with cofinality ω and cardinality at most 2^ω), then X has a coarser nowhere Urysohn topology.*

Proof. The subspace W has a coarser topology which makes it homeomorphic to Example 4.1. Because $X = (X \setminus W) \oplus W$, we can now apply Corollary 4.5. \square

If X is an extension of ω and is either not feebly compact or regular not countably compact, then X satisfies the hypothesis of Corollary 4.6.

5. Sufficient conditions via super-epoxicity

In the previous section considered topological sums $Z \oplus X$ and sought coarser topologies in which X is dense and open. In this section we focus on finding coarser connected topologies in which X is connected (or nowhere Urysohn) as well.

Our first goal in this section is to construct nowhere Urysohn spaces which are more than κ -epoxic for uncountable κ . The next example illustrates the first step of the process.

Example 5.1. Let $X = \kappa$ be a cardinal with uncountable cofinality, let \mathcal{B} be the family of bounded clopen subsets of X , let $T = \{t_\nu: \nu < \kappa\}$ be a set of cardinality κ disjoint from X , let $Y = \{\xi + 1: \xi \in \kappa\}$, and let $\mathcal{S} = \{S_\nu: \nu < \kappa\}$ partition Y so that S_ν is a cofinal subset of X with cardinality κ for all $\nu < \kappa$. We define an extension $E = X \cup T$ of X . First, X is open in E . Next, a basic open neighborhood of $t \in T$ has the form $\{t_\nu\} \cup (S_\nu \setminus B)$, where $B \in \mathcal{B}$. Finally, note that if $t_\nu \in U$, open and $t_\mu \in V$ open, then $\text{cl}_E U \cap \text{cl}_E V \cap X \neq \emptyset$ because the closed unbounded subsets of κ form a filter. This example shows that the space κ is κ -super-epoxic (see Definition 5.2).

Definition 5.2. We say that a space X is κ -super-epoxic iff it has an extension E satisfying

- (1) $|E \setminus X| = \kappa$;
- (2) $E \setminus X$ is closed and discrete in E ;
- (3) There is a base \mathcal{B} for X such that $\text{cl}_E B \subset X$ for all $B \in \mathcal{B}$;

- (4) There is a pairwise disjoint family $\{U_t: t \in E \setminus X\}$ of open subsets of E with $t \in U_t$ for all $t \in E \setminus X$;
- (5) If U and V are open subsets of E meeting $E \setminus X$, then $\text{cl}_E U \cap \text{cl}_E V \cap X$ is not empty.

Clearly, a κ -super-epoxic space is κ -epoxic. Bing's space is ω -super-epoxic. If a space W is κ -super-epoxic, then so is $W \oplus Y$ for any space Y .

Next, we show that some κ -super-epoxic spaces have coarser topologies which are both κ -super-epoxic and nowhere Urysohn.

Lemma 5.3. *Let (X, τ) be κ -super-epoxic. If X has a dense set Y of isolated points with $|Y| = \kappa$, then X has a coarser topology which is κ -super-epoxic and nowhere Urysohn.*

Proof. Let E witness that X is κ -super-epoxic. Partition $E \setminus X$ into R and T , each of cardinality κ . Set $E' = X \cup T$. We will use R to define a coarser topology σ on X and reserve T to show that (X, σ) is κ -super-epoxic. Let $r: Y \rightarrow R$ be a bijection. Define $\varphi: E \rightarrow E'$ so that $\varphi(r(y)) = y$ for $r(y) \in R$ and φ is the identity on E' . Let σ be the quotient topology determined by φ . Observe that the following condition (*) is sufficient to imply that $V \in \sigma$.

$V \in \tau$ and for all $y \in Y$, if $y \in V$, then there is $B \in \mathcal{B}$ such that $U_{r(y)} \setminus \text{cl}_\tau B \subset V$.

It is clear that (E, σ) satisfies Definition 5.2(1), (2), and (5). That (E, σ) is Hausdorff and satisfies Definition 5.2(3) and (4) will follow quickly from

Claim. *Let V and W be disjoint (possibly empty) elements of \mathcal{B} . There is a disjoint subfamily of σ , $\{V^*\} \cup \{W^*\} \cup \{U^*(t): t \in T\}$ such that $V \subset V^*$, $W \subset W^*$, and $t \in U^*(t)$ for all $t \in T$.*

We will inductively define V_n , W_n , and $U_n(t)$, $t \in T$, and then set $V^* = \bigcup_{n < \omega} V_n$, $W^* = \bigcup_{n < \omega} W_n$, and $U^*(t) = \bigcup_{n < \omega} U_n(t)$ for all $t \in T$.

Set $V_0 = V$, $W_0 = W$, and $U_0(t) = U_t \setminus (\text{cl}_E V \cup \text{cl}_E W)$.

For each $n \in \omega$, set $V_{n+1} = \bigcup \{U_{r(y)} \setminus \text{cl}_E W: y \in V_n\}$, $W_{n+1} = \bigcup \{U_{r(y)} \setminus \text{cl}_E V: y \in W_n\}$, and $U_{n+1}(t) = \bigcup \{U_{r(y)} \setminus (\text{cl}_E V \cup \text{cl}_E W): y \in U_n(t)\}$. Note that all $V_n, W_n, U_n(t)$ are in τ ; hence $V^* = \bigcup_{n < \omega} V_n$, $W^* = \bigcup_{n < \omega} W_n$, and $U^*(t) = \bigcup_{n < \omega} U_n(t)$ are in τ . Next, notice that if $V_n, W_n, U_n(t)$ are disjoint, so are $V_{n+1}, W_{n+1}, U_{n+1}(t)$ (because of condition (4) of Definition 5.2). Hence $V^* = \bigcup_{n < \omega} V_n$, $W^* = \bigcup_{n < \omega} W_n$ and $U^*(t) = \bigcup_{n < \omega} U_n(t)$ are disjoint. Finally, if $y \in Y \cap V^*$, then $y \in Y \cap V_n$ for some n , and then $U_{r(y)} \setminus \text{cl}_\tau W \subset V_{n+1} \subset V^*$. Similarly W^* and $U^*(t)$ satisfy (*).

Definition 5.2(3) and (4) are immediate from the claim. To show that (E', σ) is Hausdorff: If $x \neq y \in X$ let $V, W \in \mathcal{B}$ separate x, y and apply the claim. If $x \in X, y \in T$, let $x \in V \in \mathcal{B}$ and $W = \emptyset$. If $x \neq y \in T$, let $V = W = \emptyset$.

Towards showing that (X, σ) is nowhere Urysohn, let U and V be nonempty elements of σ . Choose $y \in U \cap Y$ and $y' \in V \cap Y$. Then $r(y) \in \varphi^{\leftarrow} U$ and $r(y') \in \varphi^{\leftarrow} V$. These

are τ -open sets, so by Definition 5.2(5) there is $X \in \mathcal{X}$ satisfying $x \in \text{cl}_\tau U \cap \text{cl}_\tau V \subset \text{cl}_\sigma U \cap \text{cl}_\sigma V$. \square

From Examples 4.1, 5.1, and Lemma 5.3 we get:

Corollary 5.4. *Every infinite cardinal κ (with the order topology) has a coarser κ -super-epoxic, nowhere Urysohn topology. A fortiori, the discrete space of cardinality κ has a coarser κ -super-epoxic, nowhere Urysohn topology.*

The following corollary of Theorem 3.3 and Corollary 5.4 is a strong form of Theorem 2(a).

Corollary 5.5. *If Z embeds in a space of weight at most $2^{|X|}$ and X is discrete, then $Z \oplus X$ has a coarser connected topology.*

We need the following lemma to prove the converse of Corollary 2.4.

Lemma 5.6. *Let κ be an uncountable cardinal and η an ordinal, $0 < \eta < \kappa^+$. Then $X = \kappa \cdot \eta$ with the usual ordinal topology is κ -super-epoxic.*

Proof. The case $\eta = 1$ and κ has uncountable cofinality is Example 5.1. If $\eta = 1$ and κ has countable cofinality, we need to choose the sets S_ν carefully because the closures of two arbitrarily chosen cofinal sets may be disjoint.

Let \mathcal{B} , T , and Y be as in Example 5.1. Let $\{\lambda_\alpha: \alpha < \text{cf } \kappa\}$ be an increasing sequence of regular cardinals converging to κ . For each $\alpha < \text{cf } \kappa$, let $\{S(\alpha, \nu): \nu < \lambda_{\alpha+1}\}$ be a disjoint family of subsets of $(\lambda_\alpha, \lambda_{\alpha+1}) \cap Y$ such that each $S(\alpha, \nu)$ is cofinal in $\lambda_{\alpha+1}$. Set $S_\nu = \bigcup \{S(\alpha, \nu): \nu < \lambda_\alpha \wedge \alpha < \text{cf } \kappa\}$.

We define the extension $E = X \cup T$ of X with basic open sets defined as in Example 5.1. Towards verifying condition 5.2(5), let $\nu, \nu' \in \kappa$ and $B, B' \in \mathcal{B}$. There is $\alpha < \text{cf } \kappa$ so that $\nu, \nu', \max B, \max B' < \lambda_\alpha$. Then $S_\nu \setminus B$ and $S_{\nu'} \setminus B'$ are cofinal in $\lambda_{\alpha+1}$, a regular uncountable cardinal. Hence their closures meet.

Next, suppose that $\eta = \zeta + 1$. Then the interval $(\kappa \cdot \zeta, \kappa \cdot \eta)$ is clopen in X and is homeomorphic to κ , which is κ -super-epoxic. Hence X is κ -super-epoxic by the remark after Definition 5.2.

Finally, suppose that η is a limit ordinal. Let $\{S_\nu: \nu < \kappa\}$ be the partition of Y defined above. For each $\nu \in \kappa$, let $S'_\nu = \{\kappa \cdot \xi + \zeta: \xi < \eta, \zeta \in S_\nu\}$. Proceed as above, except using S'_ν in place of S_ν in the definition of neighborhoods of t_ν . Towards verifying condition 5.2(5), let $\nu, \nu' \in \kappa$ and $B, B' \in \mathcal{B}$. There is $\zeta < \eta$ so that $\max B, \max B' < \kappa \cdot \zeta$. Then $\kappa \cdot \zeta \in \text{cl}_X(S_\nu \setminus B) \cap \text{cl}_X(S'_{\nu'} \setminus B')$. \square

The next theorem is the converse of Corollary 2.4.

Theorem 5.7. *Let δ be a limit ordinal. Set $\beta = \inf\{\gamma > 0: \exists \eta \ \delta = \eta + \gamma\}$ and set $\alpha = \inf\{\gamma: \gamma + \beta = \delta\}$. If $|\alpha| \leq 2^{|\beta|}$ then δ with the order topology has a coarser connected topology.*

Proof. If β is countable, then $\delta = \alpha + \beta$ has cardinality $|\alpha| \leq 2^\omega$, and this case was done in Corollary 4.6.

Otherwise let $\kappa = |\beta|$. By ordinal division, there are (unique) ordinals η and ρ satisfying $\beta = \kappa \cdot \eta + \rho$ and $0 \leq \rho < \beta$. Because β is minimal, $\rho = 0$ (and η is indecomposable). By Lemma 5.6, β is κ -super-epoxic. Because β has a dense set of isolated points, we can apply Lemma 5.3 to obtain a coarser κ -super-epoxic topology on β . Because $|\alpha| \leq 2^\kappa$, α can be embedded into a space of density κ . We finish by applying Corollary 4.5. \square

The next theorem is the converse of Lemma 2.6.

Lemma 5.8. *Let X be a compact space and D a discrete space of cardinality κ . If there is a qc-transversal $H \subseteq X$ such that $w(\text{cl}_X H) \leq 2^\kappa$, then $X \oplus D$ has a coarser connected topology.*

Proof. Let H be as hypothesized and set $\text{cl}_X H = Z'$. Then by Theorem 3.3, Z' can be embedded in a space Z of density κ . By Corollary 5.4, D has a coarser topology which is κ -super-epoxic and nowhere Urysohn. Then by Corollary 4.5, $\text{cl}_X H \oplus D$ has a coarser connected topology, say σ , in which D is dense, open, and connected. Define a topology ρ on $X \oplus D$ by $\rho = \{U: U \cap (\text{cl}_X H \oplus D) \in \sigma, U \cap X \in \tau(X)\}$. Note that:

- (1) $\tau(X \setminus \text{cl}_X H) \subseteq \rho$,
- (2) $\sigma|_D \subseteq \rho$ as $D \in \sigma$,
- (3) if $p \in U \in \sigma$ and $p \in \text{cl}_X H$, then there is some $V \in \tau(X)$ such that $V \cap \text{cl}_X H = U \cap \text{cl}_X H$ (so, it follows that $V \cup U \in \rho$),
- (4) $\tau|_{\text{cl}_X H} = \sigma|_{\text{cl}_X H}$ since $\text{cl}_H X$ is compact and hence minimal Hausdorff and σ is Hausdorff, and
- (5) for each $U \in \tau(X)$, $U \cap \text{cl}_H X \in \sigma|_{\text{cl}_X H}$ and there is some $W \in \sigma$ such that $U \cap \text{cl}_X H = W \cap \text{cl}_X H$ (so it follows that, $U \cup W \in \rho$).

Thus, we have that $\rho|_X = \tau(X)$, $\rho|_{\text{cl}_X H \oplus D} = \sigma$, and $(X + D, \rho)$ is Hausdorff. It is straightforward to show that $(X + D, \rho)$ is connected. \square

6. Extensions of ω

In this section we consider certain extensions of ω : sequential non-compact scattered (which are ω -super-epoxic); locally compact locally countable of size ω_1 (which are ω -super-epoxic if $\mathfrak{p} > \omega_1$); and, under CH, examples of Franklin–Rajagapalan spaces, one of which has a coarser connected topology and one of which doesn't (thus showing that $\mathfrak{p} > \omega_1$ is necessary in the previous result).

Lemma 6.1. *Let (X, τ) be an extension of ω and suppose there are*

- (1) *a denumerable set $Z \subset X \setminus \omega$;*
- (2) *a filter \mathcal{O} of relatively open subsets of Z such that $\bigcap_{O \in \mathcal{O}} \text{cl}_X O = \emptyset$;*
- (3) *a pairwise disjoint family $\{R(z, j): z \in Z, j \in \omega\} \subset [\omega]^\omega$ satisfying*

$$\forall z \in Z \forall j \in \omega \ z \in \text{cl } R(z, j).$$

Then (X, τ) is ω -super-epoxic.

Proof. We must find an extension $E = X \cup T$ of X and a base \mathcal{B} of X which satisfy Definition 5.2.

Let $T = \{t_j: j \in \omega\}$ be disjoint from X , where if $j \neq i$ then $t_j \neq t_i$. Let $S_j = \bigcup \{R(z, j): z \in Z\}$. Let $\mathcal{B} = \{B \in \tau: (\exists O \in \mathcal{O}) \text{cl}_X B \cap O = \emptyset\}$. A basic open neighborhood of $t \in E = X \cup T$ has the form $\{t_j\} \cup (S_j \setminus \text{cl}_X B)$ for some $B \in \mathcal{B}$. \square

Before applying Lemma 6.1, the following will be useful.

Definition 6.2. Let us recall the levels of the Cantor–Bendixson hierarchy: $X_0 = \{\text{isolated points in } X\}$; $X_\alpha = \{\text{isolated points in } X \setminus \bigcup_{\beta < \alpha} X_\beta\}$. X is *scattered* iff every point is in some X_α . If X is separable and scattered, we identify X_0 with ω .

Note that if X is locally countable and scattered, then $X = \bigcup \{X_\alpha: \alpha < \omega_1\}$. If X is locally compact and scattered, then $\bigcup \{X_\alpha: \alpha < \omega_1\}$ is sequential.

Corollary 6.3. *Let X be a sequential non-compact separable scattered space. X is ω -super-epoxic if it contains a countable set $Z \subset X \setminus X_0$ whose closure is not compact; in particular, if some X_α , ($\alpha > 0$), is countable.*

Many locally compact non-compact separable scattered spaces satisfy the hypothesis of Corollary 6.3: thin ones (i.e., all Cantor–Bendixson levels are countable), hence Kunen lines and Ostaszewski spaces; locally countable ones with only countably many Cantor–Bendixson levels (hence Ψ -like spaces).

Now we show that the statement “every locally compact, locally countable extension of ω with cardinality ω_1 has a coarser connected topology” is consistent with the usual axioms of set theory. We start with concepts that can be found in [2].

Definition 6.4. For S and T in $[\omega]^\omega$, we write $S \supset^* T$ iff $|S \setminus T| = \omega$ and $|T \setminus S| < \omega$.

Note that if $(T_\xi: \xi < \zeta)$ is a \supset^* -chain and ζ has cofinality ω , then there are $R \in [\omega]^\omega$ satisfying $T_\xi \supset^* R$ for all $\xi < \zeta$ (we say R is a lower bound for $(T_\xi: \xi < \zeta)$). The assertion “if $(T_\xi: \xi < \zeta)$ is a \supset^* -chain and ζ has cofinality ω_1 , then there are $R \in [\omega]^\omega$ satisfying $T_\xi \supset^* R$ for all $\xi < \zeta$ ” is (equivalent to) $\mathfrak{p} > \omega_1$.

We say that $\mathcal{T} = \{T_\xi: \xi < \delta\}$ is a *tower* (it has been suggested that mine shaft is more descriptive) iff (1) $\xi < \zeta$ implies that $T_\xi \supset^* T_\zeta$ and (2) there is no $R \in [\omega]^\omega$ satisfying $T_\xi \supset^* R$ for all $\xi < \delta$. It will be convenient to assume that $T_0 = \omega$.

Theorem 6.5. *Let X be a locally countable, locally compact extension of ω with $|X| = \omega_1$. If $\mathfrak{p} > \omega_1$ then X has a coarser connected topology.*

Proof. First notice that X is regular and sequential. Let $X = \omega \cup Y$ as in the hypothesis. Enumerate $Y = \{y_\xi: \xi < \omega_1\}$. If for some $\zeta < \omega_1$, $\{y_\xi: \xi < \zeta\}$ has noncompact closure, then we may use it as the Z of Lemma 5.1. If there is no such ζ , we may express Y as the union of a monotone increasing sequence of compact open sets, $Y = \bigcup \{Y_\alpha: \alpha < \omega_1\}$. Since X is regular, for each $\beta < \omega_1$ there is $T_\beta \subset \omega$ with $(Y \setminus Y_\beta) \cup T_\beta$ clopen. If for some $\beta' < \beta$ we have $|T_\beta \setminus T_{\beta'}| = \omega$, then X has an infinite closed discrete set of isolated points, and we are done by Corollary 6.2. So let us assume that if $\beta' < \beta$ then $T_{\beta'} \supset^* T_\beta$. By $\mathfrak{p} > \omega_1$, there is $R \in [\omega]^\omega$ with $T_\beta \supset^* R$ for all $\beta < \omega_1$. Then R is an infinite closed discrete set of isolated points, and we are done by Corollary 6.2. \square

Let us remark that we can replace “locally compact” with “regular” in the statement of Theorem 6.5: A regular, locally countable, feebly compact space is locally compact. Hence we invoke Corollary 6.2 if X is not feebly compact or Theorem 6.5 (as written) if X is feebly compact.

Finally we focus on Franklin–Rajagopalan spaces. For each tower $\mathcal{T} = \{T_\xi: \xi < \delta\}$, we define $X_{\mathcal{T}}$, an extension of ω . The point set of $X_{\mathcal{T}}$ is $\omega \cup \{z_\xi: 0 < \xi < \delta\}$. The points of ω are isolated. For $\xi < \zeta < \delta$ and $F \in [\omega]^{<\omega}$, define

$$B(\xi, \zeta, F) = \{z_\eta: \xi < \eta \leq \zeta\} \cup (T_\xi \setminus T_\zeta) \setminus F.$$

Let $\{B(\xi, \zeta, F): \xi < \zeta \wedge F \in [\omega]^{<\omega}\}$ be a basis at z_ζ . Note that $X_{\mathcal{T}}$ is locally compact.

The original construction in [6] worked with a descending chain of clopen subsets of $\beta\omega \setminus \omega$ and then applied Magill’s Theorem to obtain a space often called $\gamma\omega$. The equivalent construction using towers is more convenient for us; $\gamma\omega$ is the one-point compactification of $X_{\mathcal{T}}$ (see also [4, Problem 3.12.17]).

Lemma 6.6. *Assuming the Continuum Hypothesis, there are towers \mathcal{T} and \mathcal{R} such that $X_{\mathcal{T}}$ has a coarser connected topology and $X_{\mathcal{R}}$ has no coarser connected topology.*

Proof. For both constructions we need an enumeration $(A_\alpha: \alpha < \omega_1)$ of the infinite, coinfinite subsets of ω . We do the easier construction of \mathcal{R} first. Set $R_0 = \omega$. Given R_α , we define $R_{\alpha+1}$ by cases: $R_{\alpha+1} = R_\alpha \setminus A_\alpha$, if the latter is infinite, and $R_{\alpha+1} = R_\alpha \cap A_\alpha$ otherwise. If R_α has been defined for all $\alpha < \lambda$, λ a limit ordinal, then let R_λ be a lower bound of $\{R_\alpha: \alpha < \lambda\}$. After pruning to remove duplications and reindexing, $\mathcal{R} = \{R_\alpha: \alpha < \omega_1\}$ is a tower. It is straightforward to verify that $X_{\mathcal{R}}$ has exactly one free open ultrafilter—specifically, the family of open sets U satisfying $U \supset^* R_\alpha$ for some $\alpha < \omega_1$. We finish by applying [8, Fact 7]—if X has at least eight ($= 2^3$) clopen sets and at most one free open ultrafilter, then X has no coarser connected topology.

Next, we introduce the additional notation needed for the construction of \mathcal{T} . Fix $S = \{S_j: j \in J\}$ be a partition of ω into infinite pieces. Let us say that $T \subseteq \omega$ is *S-infinite* if $|T \cap S_j| = \omega$ for all $j \in J$.

We outline the construction: each T_ξ will be S -infinite. An extension $E = X_{\mathcal{T}} \cup J$, where J is denumerable, will witness that $X_{\mathcal{T}}$ is ω -super-epoxic as follows: Let \mathcal{B} will be the family of compact open subsets of $X_{\mathcal{T}}$. The disjoint open family will be $\{U_j: j \in J\}$, where $U_j = \{j\} \cup S_j$. A basic open neighborhood of $j \in J$ will be $U_j \setminus B$ for some $B \in \mathcal{B}$.

We set up machinery for the construction: Let $\chi: [\omega]^\omega \rightarrow [\omega]^\omega$ be a “split and choose” function: for each $A \in [\omega]^\omega$, let $\chi(A)$ be an infinite, coinfinite subset of A . Next, for T S -infinite and $A \in [\omega]^\omega$, let $\theta(T, A)$ be an S -infinite subset of T such that $\theta(T, A) \not\supset^* A$. (Such a θ can be defined by cases. Set $\theta(T, A) = T \setminus A$ if the latter is S -infinite. Otherwise $|T \cap A \cap S_j| = \omega$ for some j , and we can set $\theta(T, A) = T \setminus \chi(A \cap S_j)$.)

Finally, we construct \mathcal{T} . Set $T_0 = \omega$. If $T_{\omega \cdot \alpha}$ has been defined, for each $n \in \omega$ set $T_{\omega \cdot \alpha + n} = T_{\omega \cdot \alpha} \setminus \bigcup_{j < n} \chi(T_{\omega \cdot \alpha} \cap S_j)$. If T_η has been defined for all $\eta < \omega \cdot \alpha$, enumerate $\omega \cdot \alpha$ as $(\eta(m): m < \omega)$. Let T^* be a lower bound of $(T_\eta \cap S_j: \eta < \omega \cdot \alpha)$. Set

$$E(\omega \cdot \alpha, j) = \left(\bigcap_{m \leq j} T_{\eta(m)} \right) \cap T^*.$$

Then set $E(\omega \cdot \alpha) = \bigcup_{j \in \omega} E(\omega \cdot \alpha, j)$, and finally set $T_{\omega \cdot \alpha} = \theta(E(\omega \cdot \alpha), A_\alpha)$. Verify that $X_{\mathcal{T}}$ is ω -super-epoxic as suggested in the outline of the construction. \square

We leave to the reader the statement and proof of the Martin’s Axiom analogues of the results above.

7. Between ω and $\beta\omega$

Our earlier negative results had the form, X does not have a coarser connected topology because X does not have enough free open ultrafilters. Here we will present a separable space with the maximum number of free open ultrafilters that does not have a coarser connected topology. We introduce notions specific to $\beta\omega$.

For $A \subseteq \omega$, set $A^* = \{x \in \beta\omega: A \in x\}$ and $A^* = A^* \setminus \omega$. Then $\{A^*: A \subseteq \omega\}$ is a clopen base for $\beta\omega$ and $\{A^*: A \subseteq \omega\}$ is a clopen base for $\omega^* = \beta\omega \setminus \omega$.

For $p \in \beta\omega$ and $\{s_n: n \in \omega\} \subset \beta\omega$, let $p \lim s_n$ denote the unique element of $\bigcap_{A \in p} \text{cl}_{\beta\omega} \{s_n: n \in A\}$.

We say that a sequence $(s_n: n \in \omega)$ is *faithful* if $s_n \neq s_m$ for $n < m$ and that a pair of sequences $(s_n: n \in \omega)$ and $(t_n: n \in \omega)$ from $\beta\omega$ is *disjoint* if $s_n \neq t_m$ for all n and m . Let us say that $X \subseteq \beta\omega$ is *pervasive* iff for every pair $(s_n: n \in \omega)$ and $(t_n: n \in \omega)$ of faithful sequences there is $p \in \beta\omega$ such that $p \lim s_n \in X$ and $p \lim t_n \in X$. If $|\beta\omega \setminus X| < 2^c$ then X is pervasive. There are pervasive X with $|\beta\omega \setminus X| = 2^c$. In fact, we can construct a family of 2^c pairwise disjoint pervasive sets via an induction of length 2^c .

Lemma 7.1. *Let $(s_n: n \in \omega)$ and $(t_n: n \in \omega)$ be a disjoint pair of sequences. There is $M \in [\omega]^\omega$ such that*

$$\text{cl}_{\beta\omega} \{s_m: m \in M\} \cap \text{cl}_{\beta\omega} \{t_m: m \in M\} = \emptyset.$$

Proof. First, note that for all i and all $A \in [\omega]^\omega$ there is infinite $A' \subset A$ and $L \subset \omega$ so $s_i \in A^*$ and $t_i \cup \{t_j: j \in L\} \subset (\omega \setminus A)^*$. By induction, using symmetry, we construct $A_i, B_i, L_i, M_i, m(i)$ so

- (1) each $L_{i+1} \subset L_i, M_{i+1} \subset M_i, A_{i+1} \subset \omega \setminus \bigcup_{j \leq i} B_j, B_{i+1} \subset \omega \setminus \bigcup_{j \leq i} A_j,$
 - (2) each $s_{m(i)} \in A_i^*, t_{m(i)} \in B_i^*,$
 - (3) $\{s_j: j \in M_i\} \subset (\omega \setminus B_i)^*, \{t_j: j \in L_i\} \subset (\omega \setminus A_i)^*.$
- $M = \{m(i): i \in \omega\}$ is as desired.³ \square

Lemma 7.2. *If $X \subset \beta\omega$ is pervasive, then X has no coarser connected topology.*

Proof. Let σ be a topology on X coarser than the usual topology. For each $x \in X$, define

$$K(x) = \bigcap \{\text{cl}_{\beta\omega} U: x \in U \in \sigma\}. \quad (1)$$

Because σ is Hausdorff, if $x \neq y$, then $K(x) \cap K(y) = \emptyset$. Hence $K(x) \cap X = \{x\}$ for all $x \in X$. Because X is pervasive, $K(x)$ is finite.

By way of contradiction, assume that (X, σ) is connected. For each infinite, coinfinite $A \subset \omega$, there is $x \in X$ with $K(x) \cap A^* \neq \emptyset \neq K(x) \cap (\omega \setminus A)^*$. Hence, no $K(x) = \{x\}$. Thus, we can find a faithful pair $(s_n: n \in \omega)$ and $(t_n: n \in \omega)$ satisfying $\{s_n: n \in \omega\} \subset X, \{t_n: n \in \omega\} \subset \beta\omega \setminus X$, and $t_n \in K(s_n)$ for all $n \in \omega$. By the previous lemma, we may assume that

$$\text{cl}_{\beta\omega}\{s_m: m < \omega\} \cap \text{cl}_{\beta\omega}\{t_m: m < \omega\} = \emptyset.$$

By the definition of pervasive, there is $p \in \omega^*$ so that $\tilde{s} = p \lim(s_n: n \in \omega) \in X$ and $\tilde{t} = p \lim(t_n: n \in \omega) \in X$.

Since (X, σ) is Hausdorff, there are disjoint $u, v \in \sigma$ with $\tilde{s} \in u, \tilde{t} \in v$. But if $s_n \in u$ then $t_n \in u$, a contradiction. \square

It is natural to ask, especially considering Lemma 2.1(a), whether X having a coarser connected topology implies that $X(s)$ has a coarser connected topology. We use the methods presented above to answer, No.

Corollary 7.3. *If $X \subset \beta\omega$ is pervasive and I denotes the unit interval with the usual topology, then $X \oplus I$ has no coarser connected topology.*

Proof. Let σ be a topology on $X \oplus I$ coarser than the usual topology on $X \oplus I$. For each $y \in X \oplus I$, define

$$K(y) = \bigcap \{\text{cl}_{\beta\omega \oplus I} U: y \in U \in \sigma\}.$$

As in Lemma 7.2, the $K(y)$'s are pairwise disjoint and compact and $K(y) \cap (X \oplus I) = \{y\}$. For $y \in I, K(y) = (K(y) \setminus \{y\}) \oplus \{y\}$ is compact in $\beta\omega \oplus I$. As X is pervasive, $K(y)$ is finite. If $\{y \in X: K(y) \neq \{y\}\}$ is infinite, then the proof of Lemma 7.2 leads to a

³ Readers familiar with F-spaces will realize that Lemma 7.1 is an immediate consequence of $\beta\omega$ being an F-space.

contradiction. If $\{y \in X: K(y) \neq \{y\}\}$ is finite, then $\{y \in I: K(y) \neq \{y\}\}$ is infinite. There is a sequence $(s_n: n \in \omega)$ in I and $(t_n: n \in \omega)$ in $\beta\omega \setminus X$ such that $t_n \in K(s_n)$ and $\{s_n: n \in \omega\}$ converges to some point $s \in I$. Also, we can assume that the finite set $K(s)$ is disjoint from $\text{cl}_{\beta\omega}\{t_n: n \in \omega\}$. Pick a point $t \in X \cap \text{cl}_{\beta\omega}\{t_n: n \in \omega\}$. As in the proof of Lemma 7.2, s and t witness that $(X + I, \sigma)$ is not Hausdorff, a contradiction. \square

Example 7.4. A space which has a coarser connected topology, but whose semi-regularization does not.

For each $A \in [\omega]^\omega$, let $p_A \in A^*$ such that $p_A \neq p_B$ whenever $B \in [\omega]^\omega$ and $A \neq B$. Let $X = \omega^* \setminus \{p_A: A \in [\omega]^\omega\}$; X is pervasive as noted above.

Let I^+ denote the point set of I with the following topology: $U \subseteq I^+$ is open iff $p \in U$ implies there is some $V \in \tau(I)$ such that $p \in V$ and $V \cap \mathbb{Q} \subseteq U$. The space I^+ is H-closed and $I^+(s) = I$.

Consider the following topology on $X \oplus I^+$: Let $f: [\omega]^\omega \rightarrow I \setminus \mathbb{Q}$ be a 1–1 onto function. Define a coarser topology σ on $X \oplus I^+$ by $U \in \sigma$ iff $U \in \tau(X \oplus I^+)$ and if $f(A) \in U$, there is some $V \in p_A$ such that $V^* \cap X \subseteq U$. Let Z denote $(X \oplus I^+, \sigma)$.

We claim that Z is Hausdorff: Since each $A^* \cap X$ is open in Z , two distinct points in X can be separated by open sets. Because there is a base for I^+ whose sets each contain at most one irrational, two distinct points in I can be separated by open sets. Since every rational in I has a σ -neighborhood avoiding X , every point in X can be separated from every point in $I \cap \mathbb{Q}$. Finally, suppose $x \in \beta\omega$, $r \in I \setminus \mathbb{Q}$. Let $A = f^{-1}r$. There are disjoint $B \in x$, $C \in p_A$. If $x \in X$, this can be used to separate x from r . If $s = f(D)$ and $x = p_D$, this can be used to separate s from r .

We claim that Z is connected: Suppose U is a nonempty σ -clopen set. We show that $U = Z$. Note that for all $A \in [\omega]^\omega$, $f(A) \in \text{cl}_\sigma(A^* \cap X)$. So $U \cap I^+ \neq \emptyset$. Since $I^+(s) = I$, there are no nontrivial σ -clopen subsets of I^+ . So $I \subseteq U$. Hence, for each $A \in [\omega]^\omega$, $f(A) \in U$, so $U \cap A^* \neq \emptyset$. So U is dense in X , hence $U \supset X$.

We have shown that $(X \oplus I^+)$ has a coarser connected topology. But by Corollary 7.3, $(X \oplus I^+)(s) = X \oplus I$ does not have a coarser connected topology.

8. Questions

- (1) Can we replace “compact” with “minimal Hausdorff” in Lemma 2.6?
- (2) Is it consistent that all separable, locally compact, not compact, scattered spaces have a coarser connected topology?
- (3) Is there a space X which is not compact and has no coarser connected topology satisfying $|X| = \mathfrak{c}$ and $X \subset \beta\omega$?
- (4) By Theorem 2(a), for every space K there is a discrete space D such that $K \oplus D$ has a coarser connected topology. Let $\text{cct}(K)$ be the least cardinality of such a D . Assuming that K is compact, find inequalities relating $\text{cct}(K)$ and $|\{A \subseteq K: A \text{ clopen}\}|$, and find examples showing that these inequalities are best possible.

References

- [1] R.H. Bing, A connected countable Hausdorff space, *Proc. Amer. Math. Soc.* 4 (1953) 474.
- [2] E.K. van Douwen, The integers and topology, in: Kunen, Vaughan (Eds.), *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 111–168.
- [3] A. Dow, J. Porter, Cardinalities of H-closed spaces, *Topology Proc.* 7 (1982) 27–50.
- [4] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
- [5] J. Flachsmeier, Zur Theorie der H-abgeschlossenen Erweiterungen, *Math. Z.* 91 (1966) 349–381.
- [6] S.P. Franklin, M. Rajagopalan, Some examples in topology, *Trans. Amer. Math. Soc.* 155 (1971) 305–314.
- [7] K.D. Magill Jr, A note on compactifications, *Math. Z.* 94 (1966) 322–325.
- [8] J.R. Porter, Connected Hausdorff subtopologies, *Comment. Math. Univ. Carolin.* 42 (2001) 195–201.
- [9] J.R. Porter, R.G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer, Berlin, 1988.
- [10] M.G. Tkachenko, V.V. Tkachuk, V.V. Uspenskij, R.G. Wilson, In quest of weaker connected topologies, *Comment. Math. Univ. Carolin.* 37 (1996) 825–841.
- [11] P. Urysohn, Über die Metrisation der zusammenhängenden Mengen, *Math. Ann.* 94 (1925) 262–295.